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T-D Albert and K Ruhlig

Physikalisches Institut der Universität Bonn, Nußallee 12, D-53115 Bonn, Germany

E-mail: t-albert@th.physik.uni-bonn.de and ruhlig@th.physik.uni-bonn.de

Abstract
Employing a change of basis, the so-called factorizing Drinfel'd twist, we
construct polarization-free and completely symmetric creation operators for
a face type model equivalent to the Belavin model. A resolution of the nested
structure of the Bethe vectors is achieved.

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1. Introduction

The realm of integrable systems in two-dimensional statistical physics or $(1 + 1)$-dimensional
QFT has attracted much attention for a long time. Important contributions in this development
were the solution of the anisotropic $sl(2)$ Heisenberg model (XYZ model) by means of the
Bethe ansatz [1] and its subsequent reformulation in an algebraic version, the algebraic Bethe
ansatz or the so-called quantum inverse scattering method (QISM) [2]. Generalizations to
higher-rank groups, which allow us to treat models with internal degrees of freedom, were in
this context achieved in [3]. The efforts resulted in a plethora of models soluble by means of
the QISM, insofar as eigenvalues and eigenvectors of the Hamiltonian were found explicitly.
This method also stimulated the investigation of various areas in mathematical physics, such as
quantum groups, theory of knots etc. Despite the indisputable achievements of the QISM and
the rather simple action of the inverse problem operators, which can be interpreted as creation
and annihilation operators for quasiparticles and as generating functions for the conserved
quantities respectively, the study of correlation functions and form factors has proven to be
rather intricate. This is partly due to the fact that the solution of the inverse problem (expressing
the original microscopic operators by means of the operators figuring in the algebraic Bethe
ansatz) has only been achieved recently [4–6], and by the fact that the action of the quasipar-
ticle creation and annihilation operators which figure in the construction of the eigenfunctions
(Bethe wavevectors) is obscured on the level of the microscopic variables (spin raising and
lowering operators) by nonlocal effects arising from polarization clouds (compensating pairs
of local raising and lowering spin operators). In a seminal paper Maillet and Sanchez de
Santos [7] revealed an application of so-called Drinfel'd twists to obtain a basis for the $sl(2)$
XXX and XXZ model which allows us to express the creation and annihilation operators
in a completely symmetric way with the further advantage of being polarization free, that is

being built from the respective quasiclassical Gaudin operators dressed diagonally, thus suppressing non-local effects for these operators. A generalized transformation has subsequently been applied to the $sl(2)$ XYZ model [8] and been used to resolve the nested hierarchy in the Bethevectors of the $sl(n)$ XXX model [9].

In this paper we report the construction of suitable Drinfel'd twists for the $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetric Belavin model [10], which can be thought of as an n-state generalization of the $sl(2)$ XYZ model. Our results generalize the findings for the $sl(n)$ XXX model and provide a completely symmetric representation of the creation operators as well as a resolution of the hierarchical structure of the Bethe wavevectors.

The paper is organized as follows: section 2 provides a short survey of the Belavin model and its reformulation as a face-type model. Section 3 deals with the factorizing twists and the computation of the operator-valued entries of the monodromy matrix. In section 4 we discuss the Bethe wavevectors and section 5 contains the conclusions.

2. Belavin model and the corresponding IRF model

A possible n-state generalization of the eight-vertex model [1] is given by the $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetric model of Belavin [10], whose Boltzmann weights fulfil the Yang–Baxter equation (YBE) [11, 12]

$$S(z_1 - z_2)_{k_1,k_2}^{i_1,i_2} \, S(z_1 - z_3)_{j_1,k_3}^{k_1,i_3} \, S(z_2 - z_3)_{j_2,j_3}^{k_2,k_3} = S(z_2 - z_3)_{k_2,k_3}^{i_2,i_3} \, S(z_1 - z_3)_{k_1,j_3}^{i_1,k_3} \, S(z_1 - z_2)_{j_1,j_2}^{k_1,k_2} \tag{1}$$

(here and subsequently double indices k mean summation over $0, 1, \ldots, n-1$ unless stated otherwise). It can be parametrized in the following way [13][1]:

$$S(z)_{i,j}^{k,l} = \begin{cases} 0 & \text{if} \quad i+j \neq k+l \\ \dfrac{h(w)h(z)\theta^{(i-j)}(z+w)}{h(z+w)\theta^{(i-k)}(w)\theta^{(k-j)}(z)} & \text{if} \quad i+j = k+l \quad \mod n \end{cases} \tag{2}$$

with

$$h(z) = \prod_{i=0}^{n-1} \theta^{(i)}(z) \left(\prod_{i=1}^{n-1} \theta^{(i)}(0) \right)^{-1} \tag{3}$$

where $\theta^{(i)}(z) = \theta\!\left(\genfrac{}{}{0pt}{}{\frac{1}{2} - \frac{i}{n}}{\frac{1}{2}}\right)(z)$ represents the theta function of rational characteristics $1/2 - i/n, 1/2$. The theta function of characteristics $a, b \in \mathbb{R}$ is given as a Fourier series (τ is a fixed complex number in the upper half plane, and $\Lambda_\tau \equiv \mathbb{Z} + \mathbb{Z}\tau$ is the lattice generated by 1 and τ)

$$\theta\begin{pmatrix} a \\ b \end{pmatrix}(z) = \sum_{m \in \mathbb{Z}} \exp(\pi \, i\tau(m+a)z + 2\pi \, i(m+a)(z+b)) \tag{4}$$

and has zeros at $z = (\frac{1}{2} - b) + (\frac{1}{2} - a)\tau \mod \Lambda_\tau$.

The matrix (2) is unitary, $S(z)S(-z) = \mathbb{I}$, and obeys the initial condition $S(0) = P$ [15], where P is the permutation operator.

There exists a vertex–face map which transforms the Belavin model into a face type model [14], which in turn can be thought of as a multicomponent generalization of the six-vertex model.

The correspondence is given by

$$S(z_1 - z_2)M_a^\mu(z_1) \otimes M_{a+\hat\mu}^\nu(z_2) = M_{a+\hat{\nu}'}^{\mu'}(z_1) \otimes M_a^{\nu'}(z_2) R(z_1 - z_2 | a)_{\mu',\nu'}^{\mu,\nu} \tag{5}$$

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[1] The slight difference compared to [13] originates from the normalization of the S matrix.

where $R(a|z)$ has the form [16]

$$R(z|a)^{\mu,\nu}_{\mu',\nu'} = b^{\mu,\nu}(z|a)\delta_{\mu'\mu}\delta_{\nu'\nu} + c^{\mu,\nu}(z|a)\delta_{\mu'\nu}\delta_{\nu'\mu} \tag{6}$$

$$b^{\mu,\nu}(z|a) = \frac{h(z)}{h(z+w)}\frac{h(s^\nu - s^\mu + w(a-\hat{v})^{\nu,\mu})}{h(s^\nu - s^\mu + w(a-\hat{v})^{\nu,\mu}+w)}\prod_{i=0}^{n-1}\frac{g_{i\nu}(a)}{g_{i\nu}(a+\hat{\mu})}\prod_{i=0}^{n-1}\frac{g_{i\mu}(a+\hat{v})}{g_{i\mu}(a)}$$

$$c^{\mu,\nu}(z|a) = \frac{h(z+w)}{h(z)}\frac{h(s^\nu - s^\mu + w(a-\hat{v})^{\nu,\mu}+w+z)}{h(s^\nu - s^\mu + w(a-\hat{v})^{\nu,\mu}+w)} \tag{7}$$

$$g_{i\mu}(a) = \begin{cases} 1 & \text{if } i \geqslant \mu \\ h(s^0 - s^\mu + wa^{0,\mu}) & \text{if } i = 0 \\ h(s^i - s^\mu + wa^{i,\mu}-w) & \text{if } 0 < i < \mu \end{cases} \tag{8}$$

where s^μ are arbitrary complex numbers, $a \in \sum_{\mu=0}^{n-1}\mathbb{C}\Lambda_\mu$ with Λ_μ weights of the affine Lie algebra $A^{(1)}_{n-1}$, $\hat{\mu} = \Lambda_{\mu+1} - \Lambda_\mu$, $\mu = 0, 1, \ldots, n-1$ and $\Lambda_0 = \Lambda_n$. $a^{\mu,\nu}$ is given in [14] and obeys $(a - \hat{v})^{\mu,\mu} = 0$.

The intertwining vector $M^\mu_a(z)$ is given by

$$M^\mu_a(z) = {}^t(\phi^\mu_a(z)_0, \ldots, \phi^\mu_a(z)_{n-1})\prod_{i=0}^{n-1}g^{-1}_{i\mu}(a) \tag{9}$$

$$\phi^\mu_a(z)_j = \theta^{(j)}(z - nwa^{\mu,0}). \tag{10}$$

Applying a sum of threefold tensor products of functions (9) to (1) using the vertex–face map (5) and exploiting the explicit index structure of (6) we obtain a modified Yang–Baxter equation

$$R(|z_1 - z_2|a + \hat{\alpha}_3)^{\alpha_1,\alpha_2}_{\mu_1,\mu_2}R(z_1 - z_3|a)^{\mu_1,\alpha_3}_{\beta_1,\mu_3}R(z_2 - z_3|a + \hat{\beta}_1)^{\mu_2,\mu_3}_{\beta_2,\beta_3}$$
$$= R(z_2 - z_3|a)^{\alpha_2,\alpha_3}_{\mu_2,\mu_3}R(z_1 - z_3|a + \hat{\mu}_2)^{\alpha_1,\mu_3}_{\mu_1,\beta_3}R(z_1 - z_2|a)^{\mu_1,\mu_2}_{\beta_1,\beta_2}. \tag{11}$$

This can be written symbolically as (we set $R(z_i - z_j|a) = R_{ij}(a)$)

$$R_{12}(a + \hat{\sigma}_3)R_{13}(a)R_{23}(a + \hat{\sigma}_1) = R_{23}(a)R_{13}(a + \hat{\sigma}_2)R_{12}(a).$$

The monodromy matrix for a chain with N sites associated with the R matrix (6) is

$$T_{0,1\ldots N}(\lambda|a) = R_{0N}(a + \hat{\sigma}_1 + \cdots + \hat{\sigma}_{N-1})\cdots R_{02}(a + \hat{\sigma}_1)R_{01}(a) \tag{12}$$

where 0 denotes the horizontal auxiliary space with spectral parameter λ; $1, \ldots, N$ label the vertical quantum spaces (with local inhomogeneities $\{z_i\}$) whose tensor product constitutes the physical Hilbert space $\mathcal{H}_N$. An algebraic construction of eigenvalues of the monodromy matrix associated with the matrix (2) was performed in [16] and parallels the procedure for the eight-vertex model in [2]. (One has to take into account that our construction of the monodromy matrix (12) differs from the monodromy matrix in [16] by an additional transformation in the physical space.)

Subsequently we will focus on the construction of a factorizing F matrix for the R matrix (6) and pursue its consequences for the structure of the monodromy matrix (12).

3. The F basis

The factorizing F matrix for N sites (N quantum spaces), being defined by the relation $R^\sigma_{1\ldots N}(a) = F^{-1}_{\sigma(1\ldots N)}(a)F_{1\ldots N}(a)$ [7], turns out to be given by formally the same expression as found in [9] for the $sl(n)$ XXX model

$$F_{1\ldots N}(a) = \sum_{\sigma\in S_N}\sum_\alpha{}^*\prod_{i=1}^N P^{\alpha_{\sigma(i)}}_{\sigma(i)}R^{\sigma_\alpha}_{1\ldots N}(z_1, \ldots, z_N|a) \tag{13}$$

where $P_i^{\alpha_i}$ projects on the α_ith component in the ith space and the labels $\alpha_{\sigma(i)}$ satisfy the conditions

$$
\begin{aligned}
\alpha_{\sigma(i+1)} &\geqslant \alpha_{\sigma(i)} \qquad \text{if} \quad \sigma(i+1) > \sigma(i) \\
\alpha_{\sigma(i+1)} &> \alpha_{\sigma(i)} \qquad \text{if} \quad \sigma(i+1) < \sigma(i).
\end{aligned}
\tag{14}
$$

The modification of the Yang–Baxter equation (11) requires a particular rule for the handling of the parameter a in the formation of the intertwining matrix $R^\sigma(a)$ (related to the permutation σ), which can be read off from the modified composition law

$$
\begin{aligned}
R^{\sigma\sigma_i}(a) &= R_{\sigma(i),\sigma(i+1)}(\tilde{a}_i) R^\sigma(a) \\
\tilde{a}_i &= a + \hat{\sigma}_{\sigma(1)} + \cdots + \hat{\sigma}_{\sigma(i-1)}
\end{aligned}
\tag{15}
$$

where σ_i is the transposition of $i, i+1$, and σ an arbitrary permutation.

$R^\sigma(a)$ has the intertwining property

$$
R^\sigma(a) T_{0,1\ldots N}(a) = T_{0,\sigma(1)\ldots\sigma(N)}(a) R^\sigma(a + \hat{\sigma}_h)
\tag{16}
$$

where σ_h is associated with the matrix indices in the space 0.

The matrix $F_{1\ldots N}(a)$ satisfies the factorizing equation

$$
R_{1\ldots N}^\sigma(a) = F_{\sigma(1\ldots N)}^{-1}(a) F_{1\ldots N}(a).
\tag{17}
$$

A proof of the latter equation can be found in [9] and relies on the fact that

$$
P_i^\alpha P_j^\alpha R_{ij} = P_i^\alpha P_j^\alpha \mathbb{1}_{ij}.
\tag{18}
$$

The modification of the composition law induced by the parameter a is insignificant.

We will proceed by computing elements of the monodromy matrix in the new basis provided by the F matrix. The transformation law $\tilde{T}_{0,1\ldots N}(a) = F_{1\ldots N}(a) T_{0,1\ldots N}(a) F_{1\ldots N}^{-1}(a + \hat{\sigma}_h)$ is enforced by the requirement that the resulting operator is symmetric, i.e. $\tilde{T}_{0,1\ldots N}(a) = \tilde{T}_{0,\sigma(1)\ldots\sigma(N)}(a)$, which then follows from (16).

The computation of the diagonal element $T_{n-1\,n-1}(\lambda|a)$ proceeds along the same lines as that in the generalized $sl(n)$ XXX model in [9]. Let us consider the action of the matrix $F(a)$ on $T_{n-1\,n-1}(a)$:

$$
F_{1\ldots N}(a) T_{n-1\,n-1}(a) = \sum_{\sigma \in S_N} \sum_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}^{*} \prod_{i=1}^{N} P_{\sigma(i)}^{\alpha_{\sigma(i)}} R_{1\ldots N}^\sigma(a) P_0^{n-1} T_{0,1\ldots N}(a) P_0^{n-1}
$$

$$
= \sum_{\sigma \in S_N} \sum_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}^{*} \prod_{i=1}^{N} P_{\sigma(i)}^{\alpha_{\sigma(i)}} P_0^{n-1} T_{0,\sigma(1)\ldots\sigma(N)}(a) P_0^{n-1} R_{1\ldots N}^\sigma(a + \sigma_h).
\tag{19}
$$

The specialization to the entry $(n-1, n-1)$ of the auxiliary space here is achieved by the projectors P_0^{n-1}. For the second equality in (19) we have used relation (16) and the obvious fact that P_0^{n-1} commutes with $R_{1\ldots N}^\sigma(a)$. To simplify the following argument we distinguish in the sum $\sum^{*}$ cases of various multiplicities of the occurrence of the group index $n-1$:

$$
F_{1\ldots N}(a) T_{n-1\,n-1}(a) = \sum_{\sigma \in S_N} \sum_{k=0}^{N} \sum_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}^{*\prime} \prod_{j=N-k+1}^{N} P_{\sigma(j)}^{n-1} \delta_{\alpha_{\sigma(j)},n-1} \prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}}
$$

$$
\times P_0^{n-1} T_{0,\sigma(1)\ldots\sigma(N)}(a) P_0^{n-1} R_{1\ldots N}^\sigma(a + \sigma_h).
\tag{20}
$$

Let us consider the prefactor of $R_{1\ldots N}^\sigma(a + \sigma_h)$ on the rhs of equation (20) more closely. Using specific features of the R matrices we can rewrite it as follows:

$$
\prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \prod_{j=N-k+1}^{N} P_{\sigma(j)}^{n-1} P_0^{n-1} T_{0,\sigma(1)\ldots\sigma(N)}(a) P_0^{n-1}
$$

$$
= \prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \left(R_{0,\sigma(N)}\left(a + \sum_{m=1}^{N-1} \hat{\sigma}_{\sigma(m)} \right) \right)_{n-1\,n-1}^{n-1\,n-1} \cdots \left(R_{0,\sigma(N-k+1)}(a) \right)_{n-1\,n-1}^{n-1\,n-1}
$$

$$
\times P_0^{n-1} \, T_{0,\sigma(1)\ldots\sigma(N-k)}(a) \, P_0^{n-1} \prod_{j=N-k+1}^{N} P_{\sigma(j)}^{n-1}
$$

$$
= \prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \, P_0^{n-1} \, T_{0,\sigma(1)\ldots\sigma(N-k)}(a) \, P_0^{n-1} \prod_{j=N-k+1}^{N} P_{\sigma(j)}^{n-1}
$$

$$
= \prod_{i=1}^{N-k} \left(R_{0\sigma(i)}\left(a + \sum_{m=1}^{i-1} \hat{\sigma}_{\sigma(m)} \right) \right)_{n-1,\alpha_{\sigma(i)}}^{n-1,\alpha_{\sigma(i)}} \prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \prod_{j=N-k+1}^{N} P_{\sigma(j)}^{n-1} \, P_0^{n-1}. \tag{21}
$$

Inserting the rhs of (21) into equation (20) one sees that the product $\prod_i (R_{0\sigma(i)}(a + \sum_{m=1}^{i-1} \hat{\sigma}_{\sigma(m)}))_{n-1,\alpha_{\sigma(i)}}^{n-1,\alpha_{\sigma(i)}}$ creates a diagonal dressing factor for $\tilde{T}_{n-1\,n-1}(a)$ and the product of projectors applied to $R^\sigma(a + \sigma_h)$ gives $F_{1\ldots N}(a + \sigma_h)$. One obtains

$$
\tilde{T}_{n-1\,n-1}(\lambda|a) = Y_{n-1}(a) \otimes_{i=1}^{N} \mathrm{diag}\{b(\lambda - z_i), \ldots, b(\lambda - z_i), 1\} \tag{22}
$$

with the abbreviations

$$
Y_j(a) = \prod_{m=0}^{j-1} \left\{ \prod_{i=0}^{n-1} \frac{g_{ij}(\widetilde{a_m} + k_m \hat{m})}{g_{ij}(\widetilde{a_m})} \prod_{j_m=1}^{k_m} \left[\prod_{i=0}^{n-1} \frac{g_{im}(\widetilde{a_m} + (j_m - 1)\hat{m})}{g_{im}(\widetilde{a_m} + (j_m - 1)\hat{m} + \hat{j})} \right. \right.
$$
$$
\left. \left. \times \frac{h(s^m - s^j + w[\widetilde{a_m} + (j_m - 2)\hat{m}]^{m,j})}{h(s^m - s^j + w[\widetilde{a_m} + (j_m - 2)\hat{m}]^{m,j} + w)} \right] \right\} \tag{23}
$$

$$
\widetilde{a_m} = a + \sum_{i=0}^{m-1} k_i \, \hat{i}
$$

$$
b(\lambda) = \frac{h(\lambda)}{h(\lambda + w)}
$$

and k_m gives the multiplicity of the upper matrix labels $\alpha_i = m$.

To compute $T_{n-1\,n-2}(\lambda|a)$ one has to distinguish in the sum $\sum^*$ cases of various multiplicities k_{n-1} and k_{n-2} of the occurrence of group indices $n-1$ and $n-2$:

$$
F_{1\ldots N}(a) T_{n-1\,n-2}(a) = \sum_{\sigma \in S_N} \sum_{k_{n-1}=0}^{N} \sum_{k_{n-2}=0}^{N-k_{n-1}} \sum_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}^{*''} \prod_{j_{n-1}=N-k_{n-1}+1}^{N} P_{\sigma(j_{n-1})}^{n-1} \prod_{j_{n-2}=N-k_{n-1}-k_{n-2}+1}^{N-k_{n-1}}
$$

$$
\times P_{\sigma(j_{n-2})}^{n-2} \prod_{j=1}^{N-k_{n-1}-k_{n-2}} P_{\sigma(j)}^{\alpha_{\sigma(j)}} P_0^{n-1} T_{0,\sigma(1)\ldots\sigma(N)}(a) P_0^{n-2} R_{1\ldots N}^\sigma(a + \sigma_h). \tag{24}
$$

Evaluating the matrix product in $T_{0,\sigma(1)\ldots\sigma(N)}(a)$ leads to

$$
\prod_{j_{n-1}=N-k_{n-1}+1}^{N} P_{\sigma(j_{n-1})}^{n-1} \prod_{j_{n-2}=N-k_{n-1}-k_{n-2}+1}^{N-k_{n-1}} P_{\sigma(j_{n-2})}^{n-2} \prod_{j=1}^{N-k_{n-1}-k_{n-2}} P_{\sigma(j)}^{\alpha_{\sigma(j)}} P_0^{n-1} T_{0,\sigma(1)\ldots\sigma(N)}(a) P_0^{n-2}
$$

$$
= \sum_{i=N-k_{n-1}-k_{n-2}+1}^{N-k_{n-1}} (R_{0\widetilde{N-k_{n-1}}}(\hat{a}_{N-k_{n-1}}))_{n-1\,n-2}^{n-1\,n-2} \cdots (R_{0\widetilde{i+1}}(\hat{a}_{i+1}))_{n-1\,n-2}^{n-1\,n-2}
$$

$$
\times (R_{0\tilde{i}}(\hat{a}_i))_{n-2\,n-1}^{n-1\,n-2} (R_{0\widetilde{i-1}}(\hat{a}_{i-1}))_{n-2\,n-2}^{n-2\,n-2}
$$

$$
\times \cdots \times (R_{0\widetilde{N-k_{n-1}-k_{n-2}+1}}(\hat{a}_{N-k_{n-1}-k_{n-2}+1}))_{n-2\,n-2}^{n-2\,n-2}
$$

$$
\times \prod_{k=1}^{N-k_{n-1}-k_{n-2}} (R_{0\tilde{k}}(\hat{a}_k))_{n-1\,\alpha_{\tilde{k}}}^{n-2\,\alpha_{\tilde{k}}} E_{n-2,n-1}^{\tilde{i}} \prod_{j \neq i} P_{\tilde{j}}^{\alpha_j} P_{\tilde{i}}^{n-1} E_{n-1,n-2}^0
$$

$$= \sum_{i=N-k_{n-1}-k_{n-2}+1}^{N-k_{n-1}} \prod_{l=i+1}^{N-k_{n-1}} b^{n-1,n-2}(\lambda - z_{\tilde{l}}|\hat{a}_l) c^{n-2,n-1}(\lambda - z_{\tilde{i}}|\hat{a}_i)$$

$$\times \prod_{k=1}^{N-k_{n-1}-k_{n-2}} b^{n-2,\alpha_{\tilde{k}}}(\lambda - z_{\tilde{k}}|\hat{a}_k) E_{n-2,n-1}^{\tilde{i}} \prod_{j \neq i} P_{\tilde{j}}^{\alpha_j} P_{\tilde{i}}^{n-1} E_{n-1,n-2}^0 \qquad (25)$$

with the abbreviations

$$\hat{a}_i = a + \sum_{m=1}^{i-1} \hat{\sigma}_{\tilde{m}}$$

$$\tilde{i} = \sigma(i)$$

$$(E_{a,b}^i)_{\beta_i}^{\alpha_i} = \delta_{\alpha_i,a}\delta_{\beta_i,b}.$$

One notes that in the calculation the index $\alpha_{\tilde{i}}$ has changed from $n-2$ to $n-1$. As the distribution of αs is therefore no longer consistent with the conditions (14) in the sum $\sum^*$ one has to correct this fact by commuting the site $\tilde{i}$ through all higher sites $\tilde{j}$ with $\alpha_{\tilde{j}} = n-2$. So, taking into account (15), one has to insert an additional factor

$$R_{\tilde{i} \widetilde{N-k_{n-1}}}(\hat{a}_{\widetilde{N-k_{n-1}}} + \sigma_h) \cdots R_{\tilde{i} \widetilde{i+1}}(\hat{a}_{\widetilde{i+1}} + \sigma_h) \qquad (26)$$

between the projectors and $R_{1...N}^\sigma(a + \sigma_h)$ in (24). Because of equation (18) no further corrections are necessary. For the following calculation two equations are needed:

$$P_i^{n-1} P_j^{n-2} \mathbb{1}_{ij} = P_i^{n-1} P_j^{n-2} \left\{ b^{n-1,n-2}(z_i - z_j|a)^{-1} R_{ij}(a) - \frac{c^{n-2,n-1}(z_i - z_j|a)}{b^{n-1,n-2}(z_i - z_j|a)} P_{ij} \right\} \qquad (27)$$

and

$$E_{n-1,n}^i P_j^{n-1} P_{ij} = E_{n-1,n}^j P_i^{n-1} \qquad (28)$$

with $(P_{ij})_{\beta_i \beta_j}^{\alpha_i \alpha_j} = \delta_{\alpha_i,\beta_j}\delta_{\alpha_j,\beta_i}$ the permutation operator in space i and j.

Let us now concentrate on the term with $i = N - k_{n-1} - k_{n-2} + 1$ in (25) and use (27) to create the needed factor (26). Because of (28) the second term in (27) gives rise to an $E_{n-2,n-1}^{\tilde{j}}$ with $\tilde{j} \neq \tilde{i}$. So the only possibility to obtain an $E_{n-2,n-1}^{\tilde{i}}$ is to use the first term in (27). Corrections in the other terms with $j > N - k_{n-1} - k_{n-2} + 1$ cannot lead to an expression with $E_{n-2,n-1}^{\tilde{i}}$ as $\tilde{j}$ has not to be commuted with the site $\tilde{i}$. So the only term that contains $E_{n-2,n-1}^{\tilde{i}}$ after the corrections for that special $R_{1...N}^\sigma$ in (25) is

$$\prod_{l=i+1}^{N-k_{n-1}} \frac{b^{n-1,n-2}(\lambda - z_{\tilde{l}}|\hat{a}_l)}{b^{n-1,n-2}(z_{\tilde{i}} - z_{\tilde{l}}|\hat{a}_l)} c^{n-2,n-1}(\lambda - z_{\tilde{i}}|\hat{a}_i) \prod_{k=1}^{N-k_{n-1}-k_{n-2}} b^{n-2,\alpha_{\tilde{k}}}(\lambda - z_{\tilde{k}}|\hat{a}_k) E_{n-2,n-1}^{\tilde{i}}$$

$$= \prod_{l=i+1}^{N-k_{n-1}} \frac{b(\lambda - z_{\tilde{l}})}{b(z_{\tilde{i}} - z_{\tilde{l}})} c^{n-2,n-1}(\lambda - z_{\tilde{i}}|\tilde{a}_{n-2}) \prod_{k=1}^{N-k_{n-1}-k_{n-2}} b^{n-2,\alpha_{\tilde{k}}}(\lambda - z_{\tilde{k}}|\hat{a}_k) E_{n-2,n-1}^{\tilde{i}}. \qquad (29)$$

Because of the symmetry of $\tilde{T}_{0,1...N}(\lambda|a)$ all other terms have to be of the same form as (29). After taking into account the projectors the resulting expression is

$$\tilde{T}_{n-1\,n-2}(\lambda|a) = Y_{n-2}(a) \sum_{i=1}^N c^{n-2,n-1}(\lambda - z_i|\tilde{a}_{n-2}) E_{n-2,n-1}^i$$

$$\otimes_{j \neq i}^N \text{diag}\{b(\lambda - z_j), \ldots, b(\lambda - z_j), b(\lambda - z_j)b^{-1}(z_i - z_j), 1\}. \qquad (30)$$

For the calculation of $\tilde{T}_{n-1\,n-3}(\lambda|a)$ one has to distinguish the cases $n-1$, $n-2$ and $n-3$ in the sum $\sum^*$. The only difference compared to $\tilde{T}_{n-1\,n-2}(\lambda|a)$ is a term containing a product

$E^i_{n-3,n-2} E^j_{n-2,n-1}$ now showing up in the matrix product in $T_{0,\sigma(1)...\sigma N}(a)$. One again has to correct the distribution of $\alpha's$ with the analogue of the equations (27) and (28)

$$P^{\alpha_i}_i P^{\alpha_j}_j \mathbb{1}_{ij} = P^{\alpha_i}_i P^{\alpha_j}_j \left\{ b^{\alpha_i,\alpha_j}(z_i - z_j|a)^{-1} R_{ij}(a) - \frac{c^{\alpha_j,\alpha_i}(z_i - z_j|a)}{b^{\alpha_i,\alpha_j}(z_i - z_j|a)} P_{ij} \right\} \tag{31}$$

$$E^i_{a,b} P^a_j P_{ij} = E^j_{a,b} P^a_i \tag{32}$$

and also with a new relation which has to be taken into account when dealing with the term containing the product $E^i_{n-3,n-2} E^j_{n-2,n-1}$:

$$E^i_{n-3,n-1} P^{n-2}_j P_{ij} = E^i_{n-3,n-2} E^j_{n-2,n-1}. \tag{33}$$

This reasoning leads to ($b_{ik} = b(z_i - z_k)$)

$$\tilde{T}_{n-1\,n-3}(\lambda|a) = Y_{n-3}(a) \sum_{i=1}^{N} c^{n-3,n-1}(\lambda - z_i|\tilde{a}_{n-3}) E^i_{n-3,n-1}$$

$$\otimes^N_{k\neq i} \mathrm{diag}\{b(\lambda - z_k), \ldots, b(\lambda - z_k), b(\lambda - z_k)b^{-1}_{ik}, b(\lambda - z_k)b^{-1}_{ik}, 1\}$$

$$+Y_{n-3}(a) \sum_{i\neq j}^{N} f_2(n-3, n-2; n-1) E^i_{n-3,n-2} E^j_{n-2,n-1}$$

$$\otimes^N_{k\neq i,j} \mathrm{diag}\{b(\lambda - z_k), \ldots, b(\lambda - z_k), b(\lambda - z_k)b^{-1}_{ik}, b(\lambda - z_k)b^{-1}_{jk}, 1\} \tag{34}$$

with

$$f_2(n-3, n-2; n-1) = c^{n-2,n-1}(\lambda - z_j|\tilde{a}_{n-2})c^{n-3,n-2}(\lambda - z_i|\tilde{a}_{n-3})$$

$$-\frac{c^{n-2,n-1}(z_i - z_j|\tilde{a}_{n-2})}{b^{n-1,n-2}(z_i - z_j|\tilde{a}_{n-2})}b^{n-1,n-2}(\lambda - z_j|\tilde{a}_{n-2})c^{n-3,n-1}(\lambda - z_i|\tilde{a}_{n-3}) \tag{35}$$

where the second term in (35) has its origin in the second term in (31) and in (33).

Proceeding in an analogous manner one obtains in the general case

$$\tilde{T}_{n-1\,\alpha}(\lambda|a) = \sum_{k=1}^{\alpha} \sum_{i_1\neq\cdots\neq i_k} \sum_{n-1-\alpha<n_1<\cdots n_k<n-1}$$

$$\times E^{(i_1)}_{n_1,n_2} \otimes E^{(i_2)}_{n_2,n_3} \otimes \cdots \otimes E^{(i_k)}_{n_k,n-1} f_k(n_1, n_2, \ldots, n_k; n-1) Y_{n-1-\alpha}(a)$$

$$\otimes_{j\neq\{i_k\}} \mathrm{diag}\Big\{ b(\lambda - z_j), \ldots, \underbrace{b(\lambda - z_j)b^{-1}_{i_1 j}, \ldots, b(\lambda - z_j)b^{-1}_{i_1 j}}_{n_2-n_1}, \ldots,$$

$$\underbrace{b(\lambda - z_j)b^{-1}_{i_k j}, \ldots, b(\lambda - z_j)b^{-1}_{i_k j}}_{n-1-n_k}, 1 \Big\} \tag{36}$$

with $f_k(n_1, n_2, \ldots, n_k; n_0)$ being defined recursively by[2]

$$f_k(n_1, n_2, \ldots, n_k; n_0) = -\frac{c^{n_k,n_0}_{i_{k-1},i_k}}{b^{n_0,n_k}_{i_{k-1},i_k}} b^{n_0,n_k}_{0,i_k} f_{k-1}(n_1, n_2, \ldots, n_{k-1}; n_0)$$

$$+c^{n_k,n_0}_{0,i_k} f_{k-1}(n_1, n_2, \ldots, n_{k-1}; n_k) \tag{37}$$

where

$$f_1(n_1; n_0) = c^{n_1,n_0}_{0,i_{n_1}}. $$

---

[2] In the rational $sl(n)$ case this recursion relation leads directly to the result in equation (42) in [9].

Remark. As the above procedure to obtain the operators of the monodromy matrix in the F basis did not rely on the invariance of the monodromy matrix under combined $sl(n)$ rotations in the auxilliary and quantum space, the $sl(n)$ XXZ model can be treated in the same way, and we obtain expressions for the $\tilde{T}_{n-1\,\alpha}$ similar to that in (36), with the difference that the a dependence vanishes, and the parametrization of the elements of the R matrix is trigonometric instead of elliptic ($c(\lambda) = \frac{\sinh(w)}{\sinh(\lambda+w)}$, $b(\lambda) = \frac{\sinh(\lambda)}{\sinh(\lambda+w)}$).

4. Bethe wavevectors

Our presentation of the hierarchical Bethe ansatz will be rather sketchy. For details we refer the reader to [3, 16].

The operators $T_{n-1\,\alpha}$ ($\alpha < n - 1$) act as quasiparticle creation operators and satisfy the Faddeev–Zamolodchikov algebra

$$T_{n-1\,\alpha}(\lambda|a)T_{n-1\,\beta}(\mu|a+\hat{\alpha}) = T_{n-1\,\gamma}(\mu|a)T_{n-1\,\delta}(\lambda|a+\hat{\gamma})R(\lambda-\mu|a)^{\delta\,\gamma}_{\alpha\,\beta} \quad (38)$$

or in components of the R matrix

$$[T_{n-1\,\alpha}(\lambda|a),\ T_{n-1\,\alpha}(\mu|a+\hat{\alpha})] = 0 \tag{39}$$

$$T_{n-1\,\alpha}(\lambda|a)T_{n-1\,\beta}(\mu|a+\hat{\alpha}) = \frac{1}{b^{\beta\alpha}(\mu-\lambda|a)}T_{n-1\,\beta}(\mu|a)T_{n-1\,\alpha}(\lambda|a+\hat{\beta})$$

$$- \frac{c^{\beta\alpha}(\mu-\lambda|a)}{b^{\beta\alpha}(\mu-\lambda|a)}T_{n-1\,\beta}(\lambda|a)T_{n-1\,\alpha}(\mu|a+\hat{\beta}). \tag{40}$$

Inspired by [3], an ansatz for a Bethe vector Ψ_n is given in terms of a linear superposition of products of operators $T_{n-1\,\alpha}$ acting on a reference state $\Omega_N^{(n)}$:

$$\Psi_n(N; \lambda_1, \ldots, \lambda_p|a) = \sum_{\alpha_1,\ldots,\alpha_p} \Phi_{\alpha_1,\ldots,\alpha_p} T_{n-1\,\alpha_1}(\lambda_1|a)T_{n-1\,\alpha_2}(\lambda_2|a+\hat{\alpha_1})$$

$$\times \cdots \times T_{n-1\,\alpha_p}\left(\lambda_p|a+\sum_{i=1}^{n-1}\hat{\alpha}_i\right)\Omega_N^{(n)} \tag{41}$$

where the reference state $\Omega_N^{(n)}$ is constituted as an N-fold tensor product of lowest-weight states $v_n^{(i)} = {}^t(0, \ldots, 0, 1)$ in $\mathbb{C}_n^{(i)}$

$$\Omega_N = \otimes_{i=1}^N v_n^{(i)}$$

and the $\Phi_{\alpha_1,\ldots,\alpha_p}$ denote some c-number coefficients, which themselves have to be chosen such that they are components of an $sl(n-1)$ wavevector, leading to a nested structure finally giving a $sl(2)$ eigenvalue problem.

It is important to note that the reference state is invariant under the F transformation

$$F\,\Omega_N^{(n)} = \Omega_N^{(n)}$$

due to the special form of the R matrix.

We will not impose the Bethe ansatz equations for the spectral parameters $\{\lambda_i\}$, which turns the vector (41) into an eigenvector of the transfer matrix, that is we consider the Bethe wavevector being 'off shell' [17].

In what follows we want to determine the functional form of such vectors, using the explicit form of the operators relevant for the Bethe wavevectors.

In order to clarify the arguments employed in the course of the computation we will present the case of $sl(2)$ and $sl(3)$ in quite some detail. The generalization to the general case of $sl(n)$, $n > 3$ will then be rather straightforward.

For the $sl(2)$ case we have from (30)

$$\tilde{T}_{10}(\lambda|a) = \sum_{i=1}^{N} c^{0,1}(\lambda - z_i|a)\sigma_+^{(i)} \otimes_{j\neq i} \begin{pmatrix} b(\lambda - z_j)b_{ij}^{-1} & 0 \\ 0 & 1 \end{pmatrix}_{[j]} \tag{42}$$

$$\tilde{\Psi}_2(N; \lambda_1, \ldots, \lambda_p|a) = \tilde{T}_{10}(\lambda_1|a)\tilde{T}_{10}(\lambda_2|a + \hat{0}) \cdots \tilde{T}_{10}(\lambda_p|a + (p-1)\cdot\hat{0})\,\Omega_N^{(2)}$$

$$= \sum_{i_1 \neq \cdots \neq i_p} B_p^{(2)}(\lambda_1, \ldots, \lambda_p|z_{i_1}, \ldots, z_{i_p}|a)\sigma_+^{(i_1)} \cdots \sigma_+^{(i_p)}\,\Omega_N^{(2)}. \tag{43}$$

The c-number coefficient $B^{(2)}(\{\lambda_i\}|\{z_i\}|a)$ is, due to the 'diagonally dressed' spin raising operators σ_+^i in (42), of the form

$$B_p^{(2)}(\lambda_1, \ldots, \lambda_p|z_1, \ldots, z_p) = \sum_{\sigma \in S_p} \prod_{m=1}^{p} c^{0,1}(\lambda_m - z_{\sigma(m)}|a + (m-1)\cdot\hat{0})$$

$$\times \prod_{l=m+1}^{p} \frac{b(\lambda_m - z_{\sigma(l)})}{b(z_{\sigma(m)} - z_{\sigma(l)})}. \tag{44}$$

We now turn to the $sl(3)$ case. The strategy will rely on the symmetry of the wavevector under the exchange of arbitrary spectral parameters (the verification of this fact follows the same lines as that in [18] using (11) and (38)), which enables us to concentrate on a particularly simple term in the sum (41), and the repeated use of the Faddeev–Zamolodchikov algebra.

These ideas lead us to propose the following form for the Bethe wavevector:

$$\tilde{\Psi}_3(N, \lambda_1, \ldots, \lambda_{p_0}; \lambda_{p_0+1}, \ldots, \lambda_{p_0+p_1}|a)$$

$$= \sum_{\sigma \in S_{p_0}} B_{p_1}^{(2)}(\lambda_{p_0+1}, \ldots, \lambda_{p_0+p_1}|\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(p_1)}|a)$$

$$\times \prod_{k=1}^{p_1} \prod_{l=p_1+1}^{p_0} b^{1,0}(\lambda_{\sigma(k)} - \lambda_{\sigma(l)}|a + (k-1)\cdot\hat{0} + (l - p_1 - 1)\cdot\hat{1})^{-1}$$

$$\times \tilde{T}_{21}(\lambda_{\sigma(p_1+1)}|a) \cdots \tilde{T}_{21}(\lambda_{\sigma(p_0)}|a + (p_0 - p_1 - 1)\cdot\hat{1})$$

$$\times \tilde{T}_{20}(\lambda_{\sigma(1)}|a + (p_0 - p_1)\cdot\hat{1}) \cdots \tilde{T}_{20}(\lambda_{\sigma(p_1)}|a + (p_0 - p_1)\cdot\hat{1}$$

$$+ (p_1 - 1)\cdot\hat{0})\,\Omega_N^{(3)}. \tag{45}$$

Consider a special term in the sum (41) of the form (which is motivated by the fact that the associated coefficient Φ is especially simple to compute, see below)

$$\tilde{T}_{20}(\lambda_1|a)\tilde{T}_{20}(\lambda_1|a + \hat{0}) \cdots \tilde{T}_{20}(\lambda_{p_1}|a + (p_1 - 1)\cdot\hat{0})$$

$$\times \tilde{T}_{21}(\lambda_{p_1+1}|a + p_1\cdot\hat{0})\tilde{T}_{21}(\lambda_{p_1+2}|a + p_1\cdot\hat{0} + \hat{1}) \cdots \tilde{T}_{21}(\lambda_{p_0}|a + p_1\cdot\hat{0}$$

$$+ (p_0 - p_1 - 1)\cdot\hat{1})\,\Omega_N^{(3)}. \tag{46}$$

Commuting all $\tilde{T}_{20}(\lambda_1|a)$ to the right using the first term in (40) yields an additional factor

$$\prod_{x=1}^{p_1} \prod_{y=p_1+1}^{p_0} \{b^{1,0}(\lambda_y - \lambda_x|a + (x-1)\cdot\hat{0} + (y - p_1 - 1)\cdot\hat{1})\}^{-1}.$$

It has to be noted that the associated c-number coefficient $\Phi_{0\ldots01\ldots1}^{(2)}$ in (46) is not evaluated in the $sl(3)$ F basis. It can however be expressed in the form (44) as it is invariant under the action of the $sl(2)$ F matrix. This is due to the fact that it constitutes a component of the $sl(2)$ vector whose labels (a non-decreasing series of α_i with respect to the original ordering of sites i) correspond via (14) to the identity permutation in the definition of the F matrix (13).

Invoking the exchange symmetry we arrive thus at the formula (45).

The creation operators with respect to the lowest-weight state are of the form (cf (36))

$$\tilde{T}_{21}(\lambda|a) = Y_1(a) \sum_{i=1}^{N} c^{1,2}(\lambda - z_i|\tilde{a}_1) E_{1,2}^{(i)} \otimes_{j \neq i} \text{diag}\{b(\lambda - z_j), b(\lambda - z_j)b_{ij}^{-1}, 1\}_{[j]} \qquad (47)$$

$$\tilde{T}_{20}(\lambda|a) = \sum_{i=1}^{N} c^{0,2}(\lambda - z_i|\tilde{a}_0) E_{0,2}^{(i)} \otimes_{j \neq i} \text{diag}\{b(\lambda - z_j)b_{ij}^{-1}, b(\lambda - z_j)b_{ij}^{-1}, 1\}_{[j]}$$

$$+\text{terms involving } E_{1,2}^{(i)} \otimes E_{0,1}^{(j)}. \qquad (48)$$

The second term in (48) annihilates the vacuum $\Omega_N^{(3)}$, which is why we did not cite the explicit form of the prefactors accompanying these roots. The form of the creation operators permits us to further simplify the wavevector (45). Taking into account the action of the roots on the respective dressing as well as the fact that both groups of creation operators generate a factor similar to the $sl(2)$ problem, we obtain

$$\tilde{\Psi}_3(N, \lambda_1, \ldots, \lambda_{p_0}; \lambda_{p_0+1}, \ldots, \lambda_{p_0+p_1}|a) = \sum_{\sigma \in S_{p_0}} \sum_{i_1 \neq \cdots \neq i_{p_0}} \prod_{k=1}^{p_1} \prod_{l=p_1+1}^{p_0} b^{1,0}(\lambda_{\sigma(l)} - \lambda_{\sigma(k)}|a$$

$$+(k-1) \cdot \hat{0} + (l - p_1 - 1) \cdot \hat{1})^{-1} b(\lambda_{\sigma(l)} - z_{i_k})$$

$$\times B_{p_0-p_1}^{1,2}(\lambda_{\sigma(p_1+1)}, \ldots, \lambda_{\sigma(p_0)}|z_{i_{p_1+1}}, \ldots, z_{i_{p_0}}|a)$$

$$\times B_{p_1}^{0,1}(\lambda_{p_0+1}, \ldots, \lambda_{p_0+p_1}|\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(p_1)}|a)$$

$$\times B_{p_1}^{0,2}(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(p_1)}|z_{i_1}, \ldots, z_{i_{p_1}}|a + (p_0 - p_1) \cdot \hat{1})$$

$$\times E_{1,2}^{(i_{p_1+1})} \cdots E_{1,2}^{(i_{p_0})} E_{0,2}^{(i_1)} \cdots E_{0,2}^{(i_{p_1})} \Omega_N^{(3)} \qquad (49)$$

where we defined as a generalization of (44)

$$B_p^{\alpha,\beta}(\lambda_1, \ldots, \lambda_p|z_1, \ldots, z_p|a) = \sum_{\sigma \in S_p} \prod_{m=1}^{p} \{c^{\alpha,\beta}(\lambda_m - z_{\sigma(m)}|\tilde{a}_\alpha + (m-1) \cdot \hat{\alpha})$$

$$\times Y_\alpha(a + (m-1) \cdot \hat{\alpha})\} \prod_{l=m+1}^{p} \frac{b(\lambda_m - z_{\sigma(l)})}{b(z_{\sigma(m)} - z_{\sigma(l)})}. \qquad (50)$$

We are now in a position to proceed to the general $sl(n)$ case.

We start with a special term of (41), where all operators $\tilde{T}_{n-1\alpha}$ are to the left of operators $\tilde{T}_{n-1\beta}$ if $\alpha < \beta$. The associated coefficient Φ again remains invariant under the action of F, which enables us to express it in a manner similar to (44). We then use the Faddeev–Zamolodchikov algebra to reverse the order of all operators. Once again only the term containing one $E_{a,b}^i$ in the expression (36) of the respective operators $\tilde{T}_{n-1\alpha}$ contributes in this special ordering.

The wavevector can be expressed in a form similar to (49) ($\bar{\alpha} \equiv n - 2 - \alpha$)

$$\tilde{\Psi}_n(N, p_0, p_1, \ldots, p_{n-2}) = \sum_{i_1 \neq \cdots \neq i_{p_0}} B_{p_0, p_1, \ldots, p_{n-2}}^{(n)}(\lambda_1, \ldots, \lambda_{p_0+\cdots+p_{n-2}}|z_{i_1}, \ldots, z_{i_{p_0}}|a)$$

$$\times \prod_{\alpha=0}^{n-2} \prod_{j=p_{\bar{\alpha}+1}+1}^{p_{\bar{\alpha}}} E_{\alpha \, n-1}^{(i_j)} \Omega_N^{(n)} \qquad (51)$$

with the recursion relation for $B^{(n)}$

$$B_{p_0 \, p_1 \, \cdots \, p_{n-2}}^{(n)}(\lambda_1, \ldots, \lambda_{p_0+p_1+\cdots+p_{n-2}}|z_1, \ldots, z_{p_0}|a)$$

$$= \sum_{\sigma \in S_{p_0}} \prod_{\alpha=0}^{n-3} \prod_{\beta=\alpha+1}^{n-2} \prod_{k_\alpha=p_{\bar{\alpha}+1}+1}^{p_{\bar{\alpha}}} \prod_{l_\beta=p_{\bar{\beta}+1}+1}^{p_{\bar{\beta}}} b(\lambda_{\sigma(l_\beta)} - z_{k_\alpha})$$

$$
\times b^{\beta,\alpha}\left(\lambda_{\sigma(l_\beta)} - \lambda_{\sigma(k_\alpha)} | a + \sum_{m=0}^{\alpha-1}(p_{\bar{m}} - p_{\bar{m}+1}) \cdot \hat{m} + (k_\alpha - p_{\bar{\alpha}+1} - 1) \cdot \hat{\alpha}\right.
$$

$$
\left. + (l_\beta - p_{\bar{\beta}+1} - 1) \cdot \hat{\beta}\right)^{-1}
$$

$$
\times \prod_{\gamma=0}^{n-2} B_{p_{\bar\gamma}-p_{\bar\gamma+1}}^{\gamma,n-1}\left(\lambda_{\sigma(p_{\bar\gamma+1}+1)} \cdots \lambda_{\sigma(p_{\bar\gamma})} | z_{p_{\bar\gamma+1}+1} \cdots z_{p_{\bar\gamma}} | a + \sum_{m=\gamma+1}^{n-2}(p_{\bar m} - p_{\bar m+1}) \cdot \hat{m}\right)
$$

$$
\times B_{p_1\cdots p_{n-2}}^{(n-1)}(\lambda_{p_0+1} \cdots \lambda_{p_0+p_1+\cdots+p_{n-2}} | \lambda_{\sigma(1)} \cdots \lambda_{\sigma(p_1)} | a) \tag{52}
$$

which can be solved explicitly to yield

$$
B_{p_0\, p_1\, \cdots\, p_{n-2}}^{(n)}(\lambda_1, \ldots, \lambda_{p_0+p_1+\cdots+p_{n-2}} | z_1, \ldots, z_{p_0} | a)
$$

$$
= \sum_{\sigma_0\in S_{p_0}} \sum_{\sigma_1\in S_{p_1}} \cdots \sum_{\sigma_{n-3}\in S_{p_{n-3}}} \prod_{i=0}^{n-3} \prod_{\alpha_i=0}^{n-3-i} \prod_{\beta_i=\alpha_i+1}^{n-2-i} \prod_{k_{\alpha_i}=p_{\bar\alpha_i+1}+1}^{p_{\bar\alpha_i}} \prod_{l_{\beta_i}=p_{\bar\beta_i+1}+1}^{p_{\bar\beta_i}}
$$

$$
\times b(\lambda_{q_{i-1}+\sigma_i(l_{\beta_i})} - \lambda_{q_{i-2}+\sigma_{i-1}(k_{\alpha_i})}) b^{\beta_i,\alpha_i}\left(\lambda_{q_{i-1}+\sigma_i(l_{\beta_i})} - \lambda_{q_{i-1}+\sigma_i(k_{\alpha_i})} | a\right.
$$

$$
\left. + \sum_{m=0}^{\alpha_i-1}(p_{\bar m} - p_{\bar m+1}) \cdot \hat{m} + (k_{\alpha_i} - p_{\bar\alpha_i+1} - 1) \cdot \hat{\alpha}_i + (l_{\beta_i} - p_{\bar\beta_i+1} - 1) \cdot \hat{\beta}_i\right)^{-1}
$$

$$
\times \prod_{\gamma_i=0}^{n-2-i} B_{p_{\bar\gamma_i}-p_{\bar\gamma_i+1}}^{\gamma_i,n-1}\left(\lambda_{q_{i-1}+\sigma_i(p_{\bar\gamma_i+1}+1)} \cdots \lambda_{q_{i-1}+\sigma_i(p_{\bar\gamma_i})} | \lambda_{q_{i-2}+\sigma_{i-1}(p_{\bar\gamma_i+1}+1)}\right.
$$

$$
\left. \cdots \lambda_{q_{i-2}+\sigma_{i-1}(p_{\bar\gamma_i})} | a + \sum_{m=\gamma_i+1}^{n-2-i}(p_{\bar m} - p_{\bar m+1}) \cdot \hat{m}\right)
$$

$$
\times B_{p_{n-2}}^{0,1}(\lambda_{q_{n-3}+1} \cdots \lambda_{q_{n-3}+p_{n-2}} | \lambda_{q_{n-4}+\sigma_{n-3}(1)} \cdots \lambda_{q_{n-4}+\sigma_{n-3}(p_{n-2})} | a) \tag{53}
$$

where we have defined

$$
q_i = \sum_{j=0}^{i} p_j \qquad q_{-1} = 0
$$

and

$$
\lambda_{\sigma_{-1}(k)} = z_k.
$$

By expressing the $sl(n)$ wavevector (51) with the help of $sl(2)$ building blocks (53) we have achieved a resolution of the hierarchy.

Remark. Once again the arguments in this section apply to the $sl(n)$ XXZ model as well, leading to a wavevector for this model which displays the same features as (51) in connection with (53), without the a dependence and with a trigonometric parametrization.

5. Conclusion

We accomplished the construction of a factorizing F matrix for the Belavin model enabling one to construct completely symmetric creation operators, which moreover are devoid of non-local effects from polarization clouds. These operators were used to resolve the intricacies of the nested structure of Bethe wavevectors.

In contrast to the $sl(n)$ XXX model the above method does not rely on an $sl(n)$ invariance of the monodromy matrix, which renders the extraction of the elements in the grid of the

monodromy matrix much more cumbersome. The only ingredients needed in our computation are the form of the R matrix (6), i.e. its structure, unitarity and the fact that it constitutes a representation of the permutation group, and the property (18). Thus our findings directly yield the corresponding expression for the generalized $sl(n)$ XXZ model.

In view of the similarities between the generalized $sl(n)$ rational, trigonometric and elliptic model it is tempting to ask whether such a distinguished basis exists for all integrable models in two dimensions. In [19] it was shown that in every integrable two-dimensional quantum field theory there exist semi-local polarization-free generators which are localized in wedge-shaped regions of Minkowski space. It is conceivable that there is a relation between these operators and polarization-free operators in lattice spin models.

We hope that these results might prove useful for the construction of form factors starting from the microscopical level.

Acknowledgments

References

[1] Baxter R J 1973 Eight vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain I–III *Ann. Phys., NY* **76** 1–71
[2] Faddeev L D and Takhtadzhan L A 1979 The quantum version of the inverse problem and the Heisenberg XYZ-model *Russ. Math. Surv.* **34** 11–68
[3] Kulish P P and Reshetikin N Y 1981 Generalized Heisenberg ferromagnet and the Gross–Neveu model *JETP* **53** 108–14
[4] Kitanine N, Maillet J M and Terras V 1999 Form factors of the XXZ Heisenberg spin-1/2 finite chain *Nucl. Phys.* B **554** 647–78
   (Kitanine N, Maillet J M and Terras V 1998 *Preprint* math-ph/9807020)
[5] Göhmann F and Korepin V E 2000 Solution of the quantum inverse problem *J. Phys. A: Math. Gen.* **33** 1199–220
   (Göhmann F and Korepin V E 1999 *Preprint* hep-th/9910253)
[6] Maillet J M and Terras V 2000 On the quantum inverse scattering problem *Nucl. Phys.* B **575** 627–44
   (Maillet J M and Terras V 1999 *Preprint* hep-th/9911030)
[7] Maillet J M and Sanchez de Santos J 2000 Drinfel'd twists and algebraic Bethe ansatz *Am. Math. Soc. Transl. (2)* **201** 137–78
   (Maillet J M and Sanchez de Santos J 1996 *Preprint* q-alg/9612012)
[8] Albert T-D, Boos H, Flume R, Poghossian R H and Ruhlig K 2001 An F-twisted XYZ model *Lett. Math. Phys.* **53** 201–14
   (Albert T-D, Boos H, Flume R, Poghossian R H and Ruhlig K 2000 *Preprint* nlin-SI/0005023)
[9] Albert T-D, Boos H, Flume R and Ruhlig K 2000 Resolution of the nested hierarchy for rational $sl(n)$ models *J. Phys. A: Math. Gen.* **33** 4963–80
   (Albert T-D, Boos H, Flume R and Ruhlig K 2000 *Preprint* nlin-SI/0002027)
[10] Belavin A A 1981 Dynamical symmetry of integrable quantum systems *Nucl. Phys.* B **180** 189–200
[11] Bovier A 1983 Factorized S matrices and generalized Baxter models *J. Math. Phys.* **24** 631–41
[12] Cherednik I V 1983 *J. Sov. Nucl. Phys.* **36** 320–4
[13] Richey M P and Tracy C A 1986 $\mathbb{Z}_n$ Baxter model: symmetries and the Belavin parametrization *J. Stat. Phys.* **42** 311–48
[14] Jimbo M, Miwa T and Okado M 1988 Local state probabilities of solvable lattice models: an $A_{n-1}^{(1)}$ family *Nucl. Phys.* **300** 74–108
[15] Quano Y H 1994 Sklyanin algebra and integrable lattice models *Int. J. Mod. Phys.* A **9** 2245–81
[16] Zhou Y-K, Yan M-L and Hou B-Y 1988 Exact solution of the $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetric generalization of Baxter's eight-vertex model *J. Phys. A: Math. Gen.* **21** L929–34
[17] Babujian G M 1993 Off-shell Bethe ansatz equation and N point correlators in $SU(2)$ WZNW theory *Int. J. Mod. Phys.* A **26** 6981
[18] Takhtadzhyan L A 1983 Quantum inverse scattering method and algebraized matrix Bethe-ansatz *J. Sov. Math.* **23** 2470–86
[19] Borchers H-J, Buchholz D and Schroer B 2000 Polarization-free generators and the S-matrix *Preprint* hep-th/0003243